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POINTWISE CONVERGENCE AND RADIAL LIMITS OF HARMONIC FUNCTIONS

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ABSTRACT

This paper characterizes those real-valued functions on a compact set K in $\mathbb{R}^n$ that can be expressed as the pointwise limit of a sequence (h_m), where each function h_m is harmonic on some neighbourhood of K. It also characterizes those functions on the unit sphere that can arise as the radial limit function at infinity of an entire harmonic function. Both results rely on important recent work of Lukeš et al. concerning approximation of affine Baire-one functions.

1. Introduction

Let K be a compact set in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, and let $\mathcal{H}(K)$ denote the collection of all functions that are harmonic on a neighbourhood of K. Further, given $x \in K$, let $\mathcal{M}_x(K)$ denote the collection of all $\mathcal{H}(K)$-*representing measures* for x, that is, probability measures μ on K satisfying

$$h(x) = \int_K h d\mu \quad \text{for every } h \in \mathcal{H}(K).$$

The following corollary of a well-known result of Debiard and Gaveau (Theorem 1 of [5], formulated originally in terms of fine harmonicity) characterizes those functions on K that are uniform limits of functions in $\mathcal{H}(K)$.

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THEOREM A: *Let K be a compact subset of $\mathbb{R}^n$ and $f: K \to \mathbb{R}$. The following statements are equivalent:*

(a) *there is a sequence (h_m) in $\mathcal{H}(K)$ such that $h_m \to f$ uniformly on K;*

(b) *$f \in C(K)$ and*

$$f(x) = \int f d\mu \quad \text{whenever } x \in K \text{ and } \mu \in \mathcal{M}_x(K). \tag{1}$$

The first main result of this paper establishes an analogue of this theorem for *pointwise* convergence of harmonic functions. We recall that a real-valued function f on a set A is called **Baire-one** if it is the pointwise limit of some sequence of continuous functions on A. A bounded Borel function f on a compact set K that satisfies (1) will be called **$\mathcal{H}$-affine** on K. All members of $\mathcal{H}(K)$ are obviously $\mathcal{H}$-affine on K.

THEOREM 1: *Let K be a compact subset of $\mathbb{R}^n$ and $f: K \to \mathbb{R}$. The following statements are equivalent:*

(a) *there is a sequence (h_m) in $\mathcal{H}(K)$ such that $h_m \to f$ pointwise on K;*

(b) *there is an increasing sequence (K_k) of compact sets with union K such that, for each k,*
   (i) *the restriction $f|_{K_k}$ is bounded, Baire-one and $\mathcal{H}$-affine on K_k, and*
   (ii) *each bounded (connected) component of $\mathbb{R}^n \backslash K_k$ intersects $\mathbb{R}^n \backslash K$.*

Example 1: Let K be the closed unit ball in $\mathbb{R}^n$ and (B_j) be a sequence of pairwise disjoint open balls of which the union V is a dense subset of K. Then the characteristic function f valued 1 on V and 0 on $K \backslash V$ is not the pointwise limit of any sequence in $\mathcal{H}(K)$. To see this, suppose otherwise. Then condition (b) of Theorem 1 must hold. Let $x_0 \in K \backslash V$ and U be an open set containing x_0. Then U contains $\overline{B}_{j_0}$ for some j_0. For any k it follows from (b)(i) and consideration of normalized surface area measure on ∂B_{j_0} that $\overline{B}_{j_0} \not\subseteq K_k$, and then from (b)(ii) that

$$(\mathbb{R}^n \backslash K_k) \cap (K \backslash V) \cap U \supseteq (\mathbb{R}^n \backslash K_k) \cap \partial B_{j_0} \neq \emptyset.$$

Thus $\mathbb{R}^n \backslash K_k$ is dense in $K \backslash V$ for each k, and a Baire category argument then yields the contradictory conclusion that $\bigcap_k (\mathbb{R}^n \backslash K_k)$ is dense in $K \backslash V$. We note that Lukeš et al. (see Lemma 3.2 in [12]) have established directly that this particular function f is not the pointwise limit of any sequence in $\mathcal{H}(K)$.

The following observations related to Theorem 1 mainly involve the fine topology of classical potential theory, namely the coarsest topology on $\mathbb{R}^n$ that makes all superharmonic functions continuous (see Chapter 7 of [1] for an account of its properties), and the associated notion of fine harmonicity (see Fuglede [8]). Their justification may be found at the end of Section 3.

Remark 1: (i) If $f\colon K \to \mathbb{R}$ is the pointwise limit of a sequence of functions in $\mathcal{H}(K)$, then f must be finely harmonic on a finely open finely dense subset of the fine interior of K.

(ii) It follows that, if every Baire-one function $f\colon K \to \mathbb{R}$ is the pointwise limit of a sequence in $\mathcal{H}(K)$, then K has empty fine interior.

(iii) It is clear from Theorem 1 that the converse of (ii) is also true, but this is an easy corollary of work of Deny [6] and Keldyš [11].

(iv) The statement of Theorem 1 remains true if, instead, we take K to be an open subset of $\mathbb{R}^n$ and interpret $\mathcal{H}(K)$ as the collection of all harmonic functions on K.

The background to our second main result is a classical theorem of Alice Roth [13] (or see Chapter IV,§5 of Gaier [11]) characterizing those functions on the unit circle T that can be expressed as $z \mapsto \lim_{r\to\infty} g(rz)$ for some entire function g. She showed that these are precisely the Baire-one functions on T that are constant on each component arc of some relatively open dense subset of T. More recently, Boivin and Paramonov [4] have established (in particular) an analogue of Roth's result for entire harmonic functions in the plane: in this case the radial limit functions are characterized as those Baire-one functions f on T for which $f(e^{i\theta})$ is a first degree polynomial of θ on each component arc of some relatively open dense subset of T.

It is natural to look for a corresponding result in higher dimensions. Thus we now consider harmonic functions h on $\mathbb{R}^n$ such that the limit

$$f(z) = \lim_{r\to\infty} h(rz)$$

exists for each z in the unit sphere S, and seek to characterize all such "radial limit functions" $f\colon S \to \mathbb{R}$. Let δ denote the Laplace–Beltrami operator on the unit sphere S; thus the Laplacian on $\mathbb{R}^n$ can be expressed in polar co-ordinates as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\delta.$$

Using an observation of Deny and Lelong concerning radial limits of bounded harmonic functions in cones (p. 104 of [7]) and a Baire category argument, it is straightforward to observe that such functions f must satisfy $\delta f = 0$ on a relatively open dense subset of S. However, it will be seen below that, when

$n \geq 3$, this property (together with f being Baire-one) is not sufficient to ensure that f is the radial limit function of some entire harmonic function.

The characterization of radial limit functions of entire harmonic functions in higher dimensions involves a suitable notion of affineness that we now introduce. If J is a compact subset of S, and if u is a function on a relatively open subset I of S such that $J \subset I$ and $\delta u = 0$ on I, then we write $u \in \mathcal{L}(J)$. Given $z \in J$ we write $\mathcal{N}_z(J)$ for the collection of all $\mathcal{L}(J)$-representing measures for z, that is, probability measures μ on J satisfying $u(z) = \int_J u d\mu$ for every $u \in \mathcal{L}(J)$, and we define the notion of an $\mathcal{L}$-affine function on J in the natural way (cf. (1)).

THEOREM 2: *Let $f: S \to \mathbb{R}$. The following statements are equivalent:*

(a) *there is a harmonic function h on $\mathbb{R}^n$ such that $h(rz) \to f(z)$ as $r \to \infty$ for each $z \in S$;*

(b) *there is an increasing sequence (J_k) of compact sets with union S such that, for each k, the restriction $f|_{J_k}$ is bounded, Baire-one and $\mathcal{L}$-affine on J_k.*

Remark 2: (i) If $n = 2$ and condition (b) of Theorem 2 holds, it follows that $f(e^{i\theta})$ must be a first degree polynomial of θ on each component of a relatively open dense subset of T. Thus Theorem 2 simplifies to the above-mentioned result of Boivin and Paramonov in this case.

(ii) If $n \geq 3$ and there is an entire harmonic function h such that $h(rz) \to f(z)$ as $r \to \infty$ for each $z \in S$, then f must be a δ-fine solution of the Laplace–Beltrami equation on a δ-finely open δ-finely dense subset of S. (By the δ-fine topology on S we mean the fine topology associated with the Laplace–Beltrami operator δ.)

(iii) To see the essential role played by the fine topology when $n \geq 3$, we note that there exist compact sets $K \subset S$ that are nowhere dense in S yet have non-empty δ-fine interior. It follows from (ii) that, for such sets K, the (Baire-one) function

$$f(x) = \begin{cases} x_1 & \text{if } x = (x_1, \ldots, x_n) \in K \\ 0 & \text{if } x \in S \backslash K \end{cases}$$

is not the radial limit function of any entire harmonic function.

The proofs of Theorems 1 and 2, both of which rely on recent work of Lukeš et al. [12], will be given in Sections 3 and 4 following some preliminary material that is assembled in the next section.

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2. Preliminary results

For each $y \in \mathbb{R}^n$ we define

$$u_y(x) = \begin{cases} -\log\|x-y\| & \text{if } n = 2, \\ \|x-y\|^{2-n} & \text{if } n \geq 3. \end{cases}$$

If $y_0, y_1 \in \mathbb{R}^n$, then by a **path** from y_0 to y_1 we mean a continuous function $g\colon [0,1] \to \mathbb{R}^n$ such that $g(0) = y_0$ and $g(1) = y_1$. Further, by a **tract** from y_0 to y_1 we mean a connected open set containing the image of such a path.

LEMMA A: *Let $K \subset \mathbb{R}^n$ be compact. If $v \in \mathcal{H}(K)$ and $\varepsilon > 0$, then there exist points $y_1, \dots, y_m$ in $\{x : 0 < \operatorname{dist}(x, K) < \varepsilon\}$ and real numbers $\alpha_1, \dots, \alpha_m$ such that*

$$\left| v - \sum_{i=1}^{m} \alpha_i u_{y_i} \right| < \varepsilon \quad \text{on } K.$$

LEMMA B: *Let T be a tract from y_0 to y_1. If $\varepsilon > 0$ and u is harmonic on $\mathbb{R}^n \backslash \{y_0\}$, then there exists a harmonic function w on $\mathbb{R}^n \backslash \{y_1\}$ such that $|w - u| < \varepsilon$ on $\mathbb{R}^n \backslash T$.*

Both these lemmas are elementary. For Lemma A we refer to Lemma 2.6.1 of [1], or Lemma 1.8 of [10]. (It is clear from the proof that the points y_i can be chosen to be arbitrarily close to K.) Lemma B relies on the technique of pole-pushing (see Section 1.6 of [10]; cf. Lemma 2.6.3 of [1]).

Let $A \subset \mathbb{R}^n$ and $G(\cdot,\cdot)$ denote the Green function of Ω, where $\Omega = \mathbb{R}^n$ if $n \geq 3$, and Ω is some open set containing $\overline{A}$ and possessing a Green function if $n = 2$ (thus we require that $\overline{A} \neq \mathbb{R}^2$ in this case). Given $x \in A$ we define μ_x^A to be the sweeping of the Dirac measure ε_x at x onto $\Omega \backslash A$ (see Section 9.1 of [1]). Thus the potential $G\mu_x^A$ coincides with the regularized reduced function $\widehat{R}_{G(\cdot,x)}^{\Omega\backslash A}$, and $\mu_x^A \neq \varepsilon_x$ if and only if $\Omega \backslash A$ is thin at x.

THEOREM B: *Let $K \subset \mathbb{R}^n$ be compact and f be a bounded Borel function on K. Then the function $x \mapsto \int f \, d\mu_x^K$ is Borel and $\mathcal{H}$-affine on K.*

THEOREM C: *Let $K \subset \mathbb{R}^n$ be compact and $f\colon K \to \mathbb{R}$ be bounded and Baire-one. If f is $\mathcal{H}$-affine on K, then there is a bounded sequence (h_m) in $C(K)$ such that each h_m is $\mathcal{H}$-affine on K and $h_m \to f$ pointwise on K.*

Theorems B and C follow from abstract results of Lukeš et al. dealing with approximation in simplicial function spaces (see Corollary 6.2 and Theorem 6.3 in [12]), in view of work of Bliedtner and Hansen [3] concerning simpliciality in potential theory. However, since the formulation of simpliciality in [12] is superficially different from that in [3], some additional explanation is in order.

Let $\mathcal{H}$ denote the collection of continuous functions on K that are finely harmonic on the fine interior of K, and let $\mathcal{M}_x^{\mathcal{H}}$ denote the collection of all probability measures μ on K that satisfy $\int h d\mu = h(x)$ whenever $h \in \mathcal{H}$. Further, let

$$\mathcal{W} = \{\min\{h_1, \ldots, h_m\} : m \in \mathbb{N} \text{ and } h_1, \ldots, h_m \in \mathcal{H}\},$$
$$\mathcal{K} = \left\{ u \in C(K) : \int u d\mu \leq u(x) \text{ whenever } x \in K \text{ and } \mu \in \mathcal{M}_x^{\mathcal{H}} \right\},$$

and let $\mathcal{M}_x^{\mathcal{K}}$ denote the collection of all probability measures μ on K satisfying $\int u d\mu \leq u(x)$ whenever $u \in \mathcal{K}$. It is easy to see that

$$\mathcal{M}_x^{\mathcal{H}} = \mathcal{M}_x^{\mathcal{K}} \tag{2}$$

since $\mathcal{H} \subseteq \mathcal{K}$. Theorem II.3.3 and Corollary II.3.8 of [3] together assert that, if $x \in K$, then μ_x^K is the unique measure in $\mathcal{M}_x^{\mathcal{H}}$ that is minimal with respect to the partial ordering given by

$$\nu \prec \mu \quad \text{if and only if} \int w d\nu \leq \int w d\mu \quad \text{whenever } w \in \mathcal{W}.$$

Further, we can appeal to Proposition I.2.6 of [3] (with $S = \mathcal{H}$ and $S_0 = \mathcal{K}$) to see that μ_x^K is also the unique measure in $\mathcal{M}_x^{\mathcal{K}}$ that is minimal with respect to the partial ordering given by

$$\nu \prec \mu \quad \text{if and only if} \int u d\nu \leq \int u d\mu \quad \text{whenever } u \in \mathcal{K}.$$

Combining this with (2) we see that $\mathcal{H}$ is "simplicial" in the terminology of Lukeš et al. and, since $\mathcal{M}_x^{\mathcal{H}} = \mathcal{M}_x(K)$ by the Debiard–Gaveau result (Theorem A), Theorems B and C now follow, as claimed, from the cited results in [12].

We note, for future reference, that the natural analogues of Theorems A–C for $\mathcal{L}$-affine functions on S also hold. One way of seeing this is to observe that the Laplace–Beltrami operator δ on S transforms, under the stereographic projection to $\mathbb{R}^{n-1}$, to a partial differential operator that is covered by the harmonic space context of [3]. (This paper also contains a general version of Theorem A.) In fact, when $n = 3$, these analogues are implicitly included in Theorems A–C since, in this case, solutions of $\delta u = 0$ on domains in S map to harmonic functions on domains in $\mathbb{R}^2 \cup \{\infty\}$ (and conversely) under the stereographic projection. Similar reasoning shows, when $n = 3$, that S, endowed with the δ-fine topology, is homeomorphic to $\mathbb{R}^2 \cup \{\infty\}$, endowed with the fine topology of classical potential theory.

3. Proof of Theorem 1

Suppose that condition (a) of Theorem 1 holds. For each $k \in \mathbb{N}$ let

$$K_k = \{x \in K : |h_m(x)| \leq k \text{ for all } m \in \mathbb{N}\}. \tag{3}$$

Then each set K_k is compact and $\bigcup_k K_k = K$. The restriction $f|_{K_k}$ is clearly bounded and Baire-one and, by dominated convergence, is $\mathcal{H}$-affine on K_k. Further, any bounded component U of $\mathbb{R}^n \backslash K_k$ satisfies $\partial U \subseteq K_k$ and so must intersect $\mathbb{R}^n \backslash K$ in view of the maximum principle. Hence condition (b) holds.

Conversely, suppose that condition (b) holds. For each $k \in \mathbb{N}$ it follows from condition (b)(i) and Theorem C that there is a sequence $(u_{k,m})_{m \geq 1}$ in $C(K_k)$ such that each $u_{k,m}$ is $\mathcal{H}$-affine on K_k and

$$u_{k,m} \to f \quad \text{as } m \to \infty, \text{ pointwise on } K_k. \tag{4}$$

For each $k, m \in \mathbb{N}$ we define

$$F_{k,m} = \begin{cases} K_1 & \text{if } k = 1 \\ \{x \in K_k : \operatorname{dist}(x, K_{k-1}) \geq 1/m\} & \text{if } k \geq 2 \end{cases}$$

and then put

$$L_m = \bigcup_{k=1}^{m} F_{k,m} \quad (m \in \mathbb{N}).$$

For each $m \in \mathbb{N}$ we define

$$v_m(x) = u_{k,m}(x) \quad (x \in F_{k,m}; k = 1, \ldots, m). \tag{5}$$

Thus $v_m \in C(L_m)$ and v_m is $\mathcal{H}$-affine on L_m, since the compact sets $F_{1,m}, \ldots, F_{m,m}$ are pairwise disjoint. Given $m \in \mathbb{N}$ we apply first Theorem A and then Lemma A to see that there exist points $y_{m,1}, \ldots, y_{m,i_m} \in \mathbb{R}^n \backslash L_m$ and real numbers $\alpha_{m,1}, \ldots, \alpha_{m,i_m}$ such that

$$\left| v_m - \sum_{i=1}^{i_m} \alpha_{m,i} u_{y_{m,i}} \right| < 1/m \quad \text{on } L_m. \tag{6}$$

We now construct a family of tracts as follows. Let

$$A = \{y_{m,i} : m \geq 1 \text{ and } 1 \leq i \leq i_m\}.$$

Then $A \cap K_1 = \emptyset$, so $A = (\bigcup_k A_k) \cup A_\infty$, where $A_k = A \cap (K_{k+1} \backslash K_k)$ and $A_\infty = A \backslash K$. In applying Lemma A we can clearly arrange that the points $y_{m,i}$ are distinct, and that any limit points of A_k belong to K_k. For each choice of

k we can now inductively choose a countable collection of tracts $\{T_x : x \in A_k\}$ such that:

(I) T_x is a tract from x to some point $x' \in \mathbb{R}^n \backslash K$ such that $\mathbb{R}^n \backslash \overline{T}_x$ is connected,

(II) $\overline{T}_x \subset \mathbb{R}^n \backslash K_k$ (see condition (b)(ii)), and

(III) the sets $\overline{T}_x$ $(x \in A_k)$ are pairwise disjoint.

For each choice of m, i such that $y_{m,i} \in K$ we apply Lemma B to the function $\alpha_{m,i} u_{y_{m,i}}$ to see that there exists a function $w_{m,i}$, harmonic on $\mathbb{R}^n$ apart from a singularity outside K, such that

$$|w_{m,i} - \alpha_{m,i} u_{y_{m,i}}| < \frac{1}{m i_m} \quad \text{on } \mathbb{R}^n \backslash T_{y_{m,i}}.$$

On the other hand, if $y_{m,i} \in A_\infty$, then we simply define $w_{m,i} = \alpha_{m,i} u_{y_{m,i}}$ and $T_{y_{m,i}} = \emptyset$. It follows from (6) that, if we define

$$w_m = \sum_{i=1}^{i_m} w_{m,i} \quad (m \in \mathbb{N}),$$

then $w_m \in \mathcal{H}(K)$ and

$$(7) \qquad |v_m - w_m| < 2/m \quad \text{on } L_m \backslash \bigcup_{i=1}^{i_m} T_{y_{m,i}}.$$

It remains to check that, for a given point $x_0 \in K$, we have $w_m(x_0) \to f(x_0)$. Let $k_0 = \min\{k \in \mathbb{N} : x_0 \in K_k\}$ and choose $m_0 \geq k_0$ large enough so that $x_0 \in F_{k_0,m_0}$. Thus $x_0 \in F_{k_0,m} \subseteq L_m$ whenever $m \geq m_0$. Properties (II) and (III) above of our family of tracts ensure that x_0 can belong to at most one member of $\{T_x : x \in A_k\}$ when $1 \leq k \leq k_0 - 1$, and that x_0 does not lie in any of the tracts $\bigcup_{k \geq k_0} \{T_x : x \in A_k\}$. Thus x_0 belongs to at most $k_0 - 1$ of the tracts $\{T_x : x \in A\}$, and so we can choose $m_1 \geq m_0$ large enough so that

$$x_0 \notin \bigcup_{m=m_1}^{\infty} \bigcup_{i=1}^{i_m} T_{y_{m,i}}.$$

It follows from (7) that

$$|v_m(x_0) - w_m(x_0)| < 2/m \quad (m \geq m_1),$$

and hence from (5) that

$$|u_{k_0,m}(x_0) - w_m(x_0)| < 2/m \quad (m \geq m_1).$$

Thus $w_m(x_0) \to f(x_0)$ in view of (4), as required.

This completes the proof of Theorem 1.

We conclude this section by justifying Remark 1. Firstly, suppose that $f\colon K \to \mathbb{R}$ is the pointwise limit of a sequence (h_m) in $\mathcal{H}(K)$ and define K_k as in (3). Then, by Theorem 11.9 of [8], f is finely harmonic on the fine interior V_k of K_k. Further, since $\mathbb{R}^n$ endowed with the fine topology is a Baire space and $\bigcup_k K_k = K$, the set $\bigcup_k V_k$ must be finely dense in the fine interior of K. This justifies part (i) of Remark 1, and (ii) follows on considering, say, the function $f(x) = \|x\|^2$. Next, suppose that K has empty fine interior. A classical result of Deny [6] and Keldyš [11] (or see Theorem 7.9.2 of [1], or Theorem 1.1 of [10]) asserts that any member of $C(K)$ is then the uniform limit of a sequence in $\mathcal{H}(K)$. It follows immediately that any Baire-one function on K is the pointwise limit of a sequence in $\mathcal{H}(K)$, as asserted in part (iii) of Remark 1.

Finally, as stated in (iv), the proof of Theorem 1 is readily modified to deal with the case where K is, instead, an open subset of $\mathbb{R}^n$. To prove that (a) implies (b) in this case, we choose (M_k) to be an increasing sequence of compact sets with union K such that each bounded component of $\mathbb{R}^n \backslash M_k$ intersects $\mathbb{R}^n \backslash K$, and then replace (3) by

$$K_k = \{x \in M_k : |h_m(x)| \leq k \text{ for all } m \in \mathbb{N}\}.$$

For the converse we define a **path** from a point $y_0 \in K$ to the Alexandroff point $\mathcal{A}$ of K to be a continuous function $g\colon [0,1) \to K$ such that $g(0) = y_0$ and $g(t) \to \mathcal{A}$ as $t \to 1-$, and then define a **tract** from y_0 to $\mathcal{A}$ to be a connected open subset of K containing the image of such a path. The argument now proceeds as before, except that property (I) of the family of tracts constructed should now read:

(I′) T_x is a tract from x to $\mathcal{A}$ and $(K \cup \{\mathcal{A}\}) \backslash \overline{T}_x$ is connected and locally connected.

(We refer to Section 3.2 of [10] for a discussion of local connectedness in this context.)

4. Proof of Theorem 2

Suppose that condition (a) of Theorem 2 holds, and let

$$J_k = \{z \in S : |h(rz)| \leq k \text{ for all } r \geq 1\} \quad (k \in \mathbb{N}).$$

Then each set J_k is compact and $\bigcup_k J_k = S$. We now fix k and dismiss the trivial case where $J_k = S$ (whence h is constant). Clearly, the restriction $f|_{J_k}$

is bounded and Baire-one. Let z belong to the δ-fine interior V_k of J_k. We note that, for any supersolution v of the Laplace–Beltrami equation on a relatively open subset V of S, the function $x \mapsto v(x/\|x\|)$ is superharmonic on the generalized cone $\{x \in \mathbb{R}^n \backslash \{0\} : x/\|x\| \in V\}$. (This is clear for C^2 functions v, and extends to the general case by a standard approximation argument.) Thus rz belongs to the fine interior U_k of the set

$$C_k = \{x \in \mathbb{R}^n \backslash \{0\} : x/\|x\| \in J_k\}$$

for any $r > 0$. By Theorem 14.1 of [8] (concerning uniqueness for the fine Dirichlet problem) and dilation,

$$h(rz) = \int h(x) d\mu_{rz}^{C_k}(x) = \int h(rx) d\mu_z^{C_k}(x) \quad (z \in V_k).$$

Letting $r \to \infty$ we obtain

$$f(z) = \int f(\frac{x}{\|x\|}) d\mu_z^{C_k}(x) \quad (z \in V_k).$$

Assuming temporarily that

$$\int f\Big(\frac{x}{\|x\|}\Big) d\mu_z^{C_k}(x) = \int f(y) d\nu_z^{J_k}(y), \tag{8}$$

where ν_z^A denotes the obvious swept measure for the Laplace–Beltrami equation, we see that

$$f(z) = \int f d\nu_z^{J_k},$$

and so f is $\mathcal{L}$-affine on J_k in view of Theorem B. Thus condition (b) of Theorem 2 holds, subject to verification of the claim (8).

To check this, we note that (8) clearly holds when J_k is replaced by a regular relatively open subset W of S and C_k is replaced by the corresponding generalized cone, and when f is replaced by a continuous function g on ∂W. Further, if g is continuous on J_k, then we can uniformly approximate g on J_k by the difference of two continuous δ-potentials on some relatively open proper subset of S containing J_k and consider a decreasing sequence (W_m) of (regular) relatively open subsets of S such that $\bigcap_m W_m = J_k$ to deduce that (8) holds with g in place of f. The claim now follows by dominated convergence, since f is bounded and Baire-one.

Conversely, suppose that condition (b) holds. We may assume, without loss of generality, that

$$J_k \subseteq \{z = (z_1, \ldots, z_n) \in S : z_n \notin (-1/k, 0)\} \quad (k \in \mathbb{N}). \tag{9}$$

Next, let

$$\omega_+ = \{rz : r > 0, z \in S \text{ and } z_n > -1/2\}$$

and

$$\omega_- = \{rz : r > 0, z \in S \text{ and } z_n < 1/2\},$$

and define

$$a_\rho = \sup\{H_{\rho,+}(x) : x \in \omega_+ \cap S\}, \tag{10}$$

where $H_{\rho,\pm}$ is the harmonic measure of $\{x \in \omega_\pm : \|x\| = \rho \text{ or } 1/\rho\}$ in the domain $\omega_\pm \cap \{1/\rho < \|x\| < \rho\}$. It is easy to see that

$$a_\rho \to 0 \quad \text{as } \rho \to \infty. \tag{11}$$

We temporarily fix k in $\mathbb{N}$. By the $\mathcal{L}$-affine version of Theorem C there is a sequence $(u_{k,m})_{m\geq 1}$ in $C(J_k)$ and a positive number c_k such that each $u_{k,m}$ is $\mathcal{L}$-affine on J_k and $u_{k,m} \to f$ pointwise on J_k, and such that

$$|u_{k,m}| \leq c_k \quad \text{for all } m. \tag{12}$$

Further, by the $\mathcal{L}$-affine version of Theorem A, we may assume that each function $u_{k,m}$ satisfies $\delta u_{k,m} = 0$ on a neighbourhood of $\overline{I}_{k,m}$, where $I_{k,m}$ is a relatively open subset of S such that

$$J_k \subset I_{k,m} \subset \left\{z \in S : z_n \neq -\frac{1}{2k}\right\} \tag{13}$$

(see (9)). We may also assume that the sequence $(I_{k,m})_{m\geq 1}$ is decreasing. Now let h_k denote the (Perron–Wiener–Brelot) solution to the Dirichlet problem on the open set

$$\omega_k = \bigcup_{m\geq 1} \{rz : z \in I_{k,m+1} \text{ and } ((m-1)!)^4 < r < ((m+1)!)^4\}$$

with boundary data

$$g_k(x) = \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\Big(\frac{x}{\|x\|}\Big) \quad (\{(m-1)!\}^4 m^2 \leq \|x\| < (m!)^4(m+1)^2; m \geq 1).$$

We claim that

$$h_k(rz) \to f(z) \quad \text{as } r \to \infty \quad \text{for each } z \in J_k. \tag{14}$$

To see this, let $r_0 > 8$. Then there exists $m \geq 2$ such that either

$$(15) \qquad (m!)^4(m+1) < r_0 \leq (m!)^4(m+1)^3$$

or

$$(16) \qquad ((m-1)!)^4 m^3 < r_0 \leq (m!)^4(m+1).$$

Suppose firstly that (15) holds. Then the functions $x \mapsto u_{k,l}(x/\|x\|)$ $(1 \leq l \leq m+1)$ are harmonic on a neighbourhood of $\overline{U}_1$, where

$$U_1 = \{x \in \omega_k : r_0/(m+1) < \|x\| < r_0(m+1)\}.$$

We now apply the maximum principle on U_1 as follows. We have

$$g_k(x) - \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\Big(\frac{x}{\|x\|}\Big) = 0$$

on

$$\{x \in \partial\omega_k : r_0/(m+1) < \|x\| < (m!)^4(m+1)^2\},$$

and

$$\left|g_k(x) - \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\Big(\frac{x}{\|x\|}\Big)\right| = \left|\frac{1}{m+1}\sum_{l=1}^{m+1} u_{k,l}\Big(\frac{x}{\|x\|}\Big) - \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\Big(\frac{x}{\|x\|}\Big)\right| \leq \frac{2}{m+1}c_k$$

on

$$\{x \in \partial\omega_k : (m!)^4(m+1)^2 \leq \|x\| < r_0(m+1)\}$$

by (12). Further, using (12) again, we see that

$$\left|h_k(x) - \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\Big(\frac{x}{\|x\|}\Big)\right| \leq 2c_k$$

on

$$\{x \in \omega_k : \|x\| = r_0/(m+1) \text{ or } \|x\| = r_0(m+1)\}.$$

In view of (13) we can use a dilation to compare the harmonic measure of the above set in U_1 with $H_{\rho,\pm}$. Hence, combining the above estimates with (10), we see that

$$(17) \qquad \left|h_k(r_0 z) - \frac{1}{m}\sum_{l=1}^{m} u_{k,l}(z)\right| \leq 2c_k\Big(\frac{1}{m+1} + a_{m+1}\Big) \quad (z \in J_k).$$

On the other hand, if (16) holds, then we can similarly apply the maximum principle on

$$\{x \in \omega_k : r_0/m < \|x\| < r_0 m\}$$

to see that

$$\left| h_k(r_0 z) - \frac{1}{m}\sum_{l=1}^{m} u_{k,l}(z) \right| \le 2c_k a_m \quad (z \in J_k). \tag{18}$$

Combining (17), (18) and (11), and noting that

$$\frac{1}{m}\sum_{l=1}^{m} u_{k,l}(z) \to f(z) \quad \text{as } m \to \infty \quad (z \in J_k),$$

we see that the claim (14) is established.

We next define the open sets

$$\Omega_k = \begin{cases} \{rz \in \omega_1 : \operatorname{dist}(z, J_1) < \frac{1}{2r}\} & (k = 1), \\ \{rz \in \omega_k : \operatorname{dist}(z, J_k) < \frac{1}{2r} \text{ and } \operatorname{dist}(z, J_{k-1}) > \frac{1}{2r}\} & (k \ge 2), \end{cases}$$

where $z \in S$ and $r > 0$, and the closed sets

$$E_k = \begin{cases} \{rz : z \in J_1 \text{ and } r \ge 1\} & (k = 1), \\ \{rz : z \in J_k, \operatorname{dist}(z, J_{k-1}) \ge 1/r \text{ and } r \ge k\} & (k \ge 2). \end{cases}$$

The sets Ω_k are clearly pairwise disjoint, the set $E = \bigcup_k E_k$ is closed, and $E_k \subset \Omega_k$ for each k. We can therefore define a harmonic function v on a neighbourhood of E by defining $v = h_k$ on Ω_k for each k. Further, it is easy to see that the set $(\mathbb{R}^n \cup \{\infty\})\backslash E$ is connected and locally connected. This allows us to apply an approximation result of Armitage and Goldstein [2] (or see Corollary 5.10 of [10]) to obtain the existence of an entire harmonic function h such that

$$|v(x) - h(x)| < 1/\|x\| \quad \text{on } E.$$

Now let $z \in S$. We define $k_0 = \min\{k : z \in J_k\}$ and then

$$r_0 = \begin{cases} 1 & \text{if } k_0 = 1, \\ \{\operatorname{dist}(z, J_{k_0 - 1})\}^{-1} & \text{if } k_0 \ge 2. \end{cases}$$

For $r \ge \max\{r_0, k_0\}$ we thus have $rz \in E_{k_0}$ and so

$$\begin{aligned} |f(z) - h(rz)| &\le |f(z) - v(rz)| + 1/r \\ &= |f(z) - h_{k_0}(rz)| + 1/r \\ &\to 0 \quad \text{as } r \to \infty, \end{aligned}$$

in view of (14).

This completes the proof of Theorem 2.

Part (ii) of Remark 2 follows from Theorem 2, the $\mathcal{L}$-affine version of Theorem C and Theorem 11.9 of [8], in view of the fact that S, endowed with the δ-fine topology, is a Baire space. The other two parts are straightforward.

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